On the Mathematical Theory of Two Star-drifts, and on the Systematic Motions of Zodiacal Stars.

By A. S. Eddington, B.A., M.Sc.

In a former paper,* the distribution of the proper motions of the stars was discussed from the point of view of the hypothesis of two drifts of stars. The method of analysis consisted in finding by trial and error the constants of two drifts which would lead to a distribution of proper motions (as regards direction) agreeing as nearly as possible with the observed distribution. There are evident disadvantages in a method of this kind; the difficulty of simultaneously adjusting five disposable constants, so as to obtain the best agreement with observation, is considerable; there is room for bias in deciding what is the best agreement with observation; and the method is inapplicable to the discussion of a series of proper motions of stars sparsely distributed over a large region of the sky, such as the Bradley proper motions.

It is the object of the first part of this paper (*Mathematical Theory*) to obtain from the observational data direct equations to determine the constants of the drifts. In the second part the theory is applied to the analysis of the proper motions of the Catalogue of Zodiacal Stars (*Astronomical Papers of the American Ephemeris*, vol. viii., part iii.); this is an example of a series of proper motions which, while sufficiently numerous to afford a good determination of the velocities of the drifts, are much too sparsely distributed to be analysed by the method of trial and error.

Mathematical Theory.

I have defined a "drift" to be a system of stars in which the motions of the individual stars relative to the mean of the system are haphazard; the whole system has in general a drift-motion relative to the Sun. For the sake of precision, the distribution of the haphazard or "peculiar motions" in the ideal drift is taken to be according to Maxwell's law.

In "Systematic Motions," p. 36, the law of distribution of the proper motions in the different directions was discussed for a system of stars forming a drift thus defined; the analysis may be briefly recapitulated here.

A small region of the sky is considered, and motions in the line of sight are neglected, so that the problem is two-dimensional.

Let the number of stars having component peculiar (linear) velocities between the limits (u, v) and $(u+du, v+dv)$ be

$$\frac{n_1 h^2}{\pi} e^{-h^2(u^2+v^2)} du dv \quad . \quad . \quad . \quad . \quad (1)$$

in accordance with Maxwell's law.

* "The Systematic Motions of the Stars," *Monthly Notices*, vol. lxvii. p. 34.

n_1 is the total number of stars of the drift.

h a constant inversely proportional to the mean peculiar speed of the stars.

Let v_1 be the velocity of the drift.

θ_1 the inclination of the direction of this drift-velocity to the axis of x.

r, θ the amount and direction of the resultant linear velocity of a star (*i.e.* drift velocity and peculiar velocity compounded).

$\rho\, d\theta$ the number of stars having proper motions in directions inclined to Ox between θ and $\theta + d\theta$.

Then
$$u^2 + v^2 = r^2 + v_1^2 - 2v_1 r \cos(\theta - \theta_1)$$
$$dudv = rdrd\theta$$

and hence
$$\rho d\theta = \frac{n_1 h^2}{\pi} d\theta \int_0^\infty e^{-h^2\left\{r^2 + v_1^2 - 2v_1 r \cos(\theta - \theta_1)\right\}} rdr \quad . \quad (2)$$

By means of the substitution
$$x = hr - hv_1 \cos(\theta - \theta_1)$$

this expression can easily be reduced to
$$\rho = \frac{n_1 h^2}{\pi} e^{-h^2 v_1^2} \left\{ \tfrac{1}{2} + \tau e^{\tau^2} \int_{-\tau}^{\infty} e^{-x^2} dx \right\} \quad . \quad . \quad . \quad (3)$$

where
$$\tau = hv_1 \cos(\theta - \theta_1)$$

or setting
$$f(\tau) = \tfrac{1}{2} + \tau e^{\tau^2} \int_{-\tau}^{\infty} e^{-x^2} dx$$
$$\rho = \frac{n_1}{\pi} e^{-h^2 v_1^2} f\big(hv_1 \cos(\theta - \theta_1)\big) \quad . \quad . \quad . \quad (4)$$

The function f was tabulated in "Systematic Motions," p. 36; a factor $\frac{2}{\sqrt{\pi}}$ was, however, inserted in defining it there, which it has been more convenient to drop now.

Consider now a series of proper motions of stars belonging to two drifts; distinguishing the corresponding constants by suffixes 1 and 2, we shall have

$$\rho = \frac{n_1}{\pi} e^{-h^2 v_1^2} f(hv_1 \cos \overline{\theta - \theta_1}) + \frac{n_2}{\pi} e^{-h^2 v_2^2} f(hv_2 \cos \overline{\theta - \theta_2}) \quad . \quad (5)$$

If the observed proper motions are considered to correspond to two drifts, a series of equations of condition will be obtained by giving various values to θ in (5), the corresponding values of ρ being derived from observation. From these the six unknowns, n_1, n_2, v_1, v_2, θ_1, θ_2 are to be derived; these, however, may be at

once reduced to five unknowns, for $n_1 + n_2$ is equal to the total number of stars.

It is of course impossible, with non-linear equations of condition such as these, rigorously to determine the values of the constants which shall make the sum of the squares of the residuals a minimum. In order to obtain what may serve, as it were, for normal equations, I have expanded each side of equation (5) in a Fourier series, and have equated the first few coefficients on each side until sufficient equations were thus formed. It is clear that, at least if the observed distribution really does correspond to two drifts, a good agreement between the observed and theoretical distributions will have been obtained when the Fourier series representing them agree as far as the coefficients of $\cos 3\theta$ and $\sin 3\theta$, and it is not likely that the agreement could be much further improved. Discordances represented by differences in the coefficients of higher harmonics are likely to be of an accidental and unimportant character.

In practice, the process of equating Fourier coefficients is equivalent to equating the sums $\Sigma \cos\theta$, $\Sigma \sin\theta$, $\Sigma \cos 2\theta$, $\Sigma \sin 2\theta$, etc., for the observed proper motions, to the corresponding integrals for the theoretical distributions. These integrals are evaluated in the note at the end of this paper. We find there that for a single drift (n_1, v_1 θ_1),

$$\left.\begin{aligned}
\int_0^{2\pi} \rho \cos(\theta - \theta_1) d\theta &= n_1 \frac{\sqrt{\pi}}{2} h v_1 e^{-\frac{1}{2}h^2 v_1^2} \left\{ J_0(\iota\tfrac{1}{2}h^2 v_1^2) - \iota J_1(\iota\tfrac{1}{2}h^2 v_1^2) \right\} \\
&= n_1 \,|\, C(hv_1) \,| \qquad \text{say} \\
\int_0^{2\pi} \rho \cos 2(\theta - \theta_1) d\theta &= n_1 \left(1 - \frac{1 - e^{-h^2 v_1^2}}{h^2 v_1^2} \right) = n_1 \,|\, D(hv_1) \,| \qquad \text{say} \\
\int_0^{2\pi} \rho \cos 3(\theta - \theta_1) d\theta &= n_1 \frac{\sqrt{\pi}}{2} h v_1 e^{-\frac{1}{2}h^2 v_1^2} \left\{ J_0(\iota\tfrac{1}{2}h^2 v_1^2) - \iota J_1(\iota\tfrac{1}{2}h^2 v_1^2) \left(1 - \frac{4}{h^2 v_1^2} \right) \right\} \\
&= n_1 \,|\, E(hv_1) \,| \qquad \text{say}
\end{aligned}\right\} \quad (6)$$

and evidently from the symmetry of the drift

$$\int_0^{2\pi} \rho \sin(\theta - \theta_1) d\theta = \int_0^{2\pi} \rho \sin 2(\theta - \theta_1) d\theta = \int_0^{2\pi} \rho \sin 3(\theta - \theta_1) d\theta = 0 \quad . \quad . \quad (7)$$

A table of the functions here introduced is given later (see Table I.).

It is here convenient to introduce complex quantities into the analysis, which will be denoted always by capital letters.

Let

$$\begin{aligned}
V_1 &= v_1 e^{\iota\theta_1} \\
C_1 &= C(hV_1) = |\, C(hv_1) \,|\, e^{\iota\theta_1} \\
D_1 &= D(hV_1) = |\, D(hv_1) \,|\, e^{2\iota\theta_1} \\
E_1 &= E(hV_1) = |\, E(hv_1) \,|\, e^{3\iota\theta_1}
\end{aligned} \quad . \quad . \quad . \quad (8)$$

The moduli have already been defined by equations (6).

Still considering one drift only

$$\int_0^{2\pi}\rho e^{\iota\theta}d\theta = e^{\iota\theta_1}\int_0^{2\pi}\rho e^{\iota(\theta-\theta_1)}d\theta$$

$$= e^{\iota\theta_1}\int_0^{2\pi}\rho\cos(\theta-\theta_1)d\theta + \iota e^{\iota\theta_1}\int_0^{2\pi}\rho\sin(\theta-\theta_1)d\theta$$

$$= e^{\iota\theta_1}n_1\,|\,C(hv_1)\,|.$$

Thus

$$\left.\begin{aligned}\int_0^{2\pi}\rho e^{\iota\theta}d\theta &= n_1C_1\\ \text{Similarly}\qquad \int_0^{2\pi}\rho e^{2\iota\theta}d\theta &= n_1D_1\\ \int_0^{2\pi}\rho e^{3\iota\theta}d\theta &= n_1E_1\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (9)$$

Return now to the consideration of two drifts. Knowing for every star the apparent direction θ (measured from an arbitrary initial line) of its observed motion, we can calculate $\Sigma\cos\theta$, $\Sigma\sin\theta$, $\Sigma\cos 2\theta$, etc. for all the stars.

Let

$$L = \frac{1}{n}\Sigma e^{\iota\theta} = \frac{1}{n}\left(\Sigma\cos\theta + \iota\Sigma\sin\theta\right)$$

$$M = \frac{1}{n}\Sigma e^{2\iota\theta} = \frac{1}{n}\left(\Sigma\cos 2\theta + \iota\Sigma\sin 2\theta\right)$$

$$N = \frac{1}{n}\Sigma e^{3\iota\theta} = \frac{1}{n}\left(\Sigma\cos 3\theta + \iota\Sigma\sin 3\theta\right)$$

where n is the total number of stars discussed.

And let $\frac{n_1}{n} = \frac{1}{2}(1+\alpha)$

so that $\frac{n_2}{n} = \frac{1}{2}(1-\alpha)$ since $n = n_1 + n_2$.

The equation of the observed and theoretical values of $\Sigma e^{\iota\theta}$, $\Sigma e^{2\iota\theta}$, and $\Sigma e^{3\iota\theta}$, in the case of two drifts, therefore leads to

$$\left.\begin{aligned}2L &= (1+\alpha)C_1 + (1-\alpha)C_2\\ 2M &= (1+\alpha)D_1 + (1-\alpha)D_2\\ 2N &= (1+\alpha)E_1 + (1-\alpha)E_2\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (10)$$

To solve these equations write

$$\left.\begin{aligned}C &= \zeta P\\ D &= \zeta P^2\\ E &= \frac{1}{\gamma}\zeta P^3\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (11)$$

Since the arguments of C_1, D_1, and E_1 are θ_1, $2\theta_1$, and $3\theta_1$, ζ and γ are real, and the argument of P_1 is θ_1. Further, it is found that

for the range of values of hV (up to about $|hV| = 1{\cdot}8$) occurring in the actual drifts, γ may be assumed constant and equal to $1{\cdot}163$. In Table I. are given the values of ζ and the moduli of C, D, E, δE and P for various values of hV. The values of $\delta E = E - \frac{1}{1{\cdot}163}\zeta P^3$ are given in order to show the accuracy of the approximation made.

TABLE I.

hV.	P.	ζ.	C.	D.	E.	δE.
0·0	·0000	1·585	·0000	·0000	·0000	·000
0·1	·0566	1·573	·0884	·0050	·0002	·000
0·2	·1124	1·561	·1755	·0197	·0017	·000
0·3	·1680	1·548	·2600	·0437	·0057	− ·001
0·4	·2227	1·531	·3409	·0759	·0134	·001
0·5	·2762	1·510	·4171	·1152	·0253	·002
0·6	·3284	1·486	·4879	·1602	·0420	·003
0·7	·3789	1·459	·5528	·2094		·005
0·8	·4275	1·431	·6115	·2614	·0901	·006
0·9	·4739	1·401	·6640	·3147		·007
1·0	·5179	1·371	·7103	·3679	·1558	·008
1·1	·5595	1·342	·7507	·4200		·008
1·2	·5984	1·313	·7856	·4701	·2337	·008
1·3	·6345	1·285	·8156	·5175		·007
1·4	·6678	1·259	·8410	·5617	·3172	·005
1·5	·6984	1·235	·8626	·6024	·3590	− ·003
1·6	·7262	1·213	·8807	·6396	·4000	·000
1·7	·7514	1·192	·8959	·6732		+ ·004
1·8	·7742	1·174	·9087	·7035	·4774	·009
1·9	·7946	1·157	·9194	·7305		·014
2·0	·8128	1·142	·9284	·7546	·5468	+ ·019

With this substitution, (10) becomes

$$2L = (1 + \alpha)\,\zeta_1\,P_1 + (1 - \alpha)\,\zeta_2\,P_2\,;\ \text{etc.}$$

Now let

$$\left.\begin{aligned}(1+\alpha)\zeta_1 &= (1+\beta)k\\ (1-\alpha)\zeta_2 &= (1-\beta)k\end{aligned}\right\} \quad . \quad . \quad . \quad (12)$$

so that $$k = \tfrac{1}{2}(1+\alpha)\zeta_1 + \tfrac{1}{2}(1-\alpha)\zeta_2 \quad . \quad . \quad . \quad (13)$$

We obtain finally

$$2L/k = (1+\beta)P_1 + (1-\beta)P_2 \quad . \quad . \quad . \quad (14)$$
$$2M/k = (1+\beta)P_1^2 + (1-\beta)P_2^2 \quad . \quad . \quad (15)$$
$$2N\gamma/k = (1+\beta)P_1^3 + (1-\beta)P_2^3 \quad . \quad . \quad (16)$$

It will be seen from (13) that k is a weighted mean between ζ_1 and ζ_2, the weights being proportional to the number of stars in the two drifts. Now looking at Table I. we see that ζ_1 and ζ_2 vary within fairly narrow limits; actually the values of hv_1 and hv_2 in the different parts of the sky are such that, except in a few unusual cases, k will lie between 1·35 and 1·45. For a first approximation, we may either calculate a provisional value of k from the roughly known speeds of the drifts, or we may assume for it the value 1·40, which will certainly be not far from the truth. A second approximation may afterwards be made if desired, using the value of k deduced from the results of the first approximation; in practice, however, this is usually quite unnecessary.

Thus k may be assumed to be known, and the solution of (14), (15), and (16) is then easy. We find

$$\frac{4}{k^2}(\mathrm{N}\gamma k - \mathrm{LM}) = (1-\beta^2)(\mathrm{P}_1-\mathrm{P}_2)^2(\mathrm{P}_1+\mathrm{P}_2) \quad . \quad (17)$$

$$\frac{4}{k^2}(\mathrm{M}k - \mathrm{L}^2) = (1-\beta^2)(\mathrm{P}_1-\mathrm{P}_2)^2 \quad . \quad . \quad . \quad (18)$$

therefore
$$\frac{\mathrm{N}\gamma k - \mathrm{LM}}{\mathrm{M}k-\mathrm{L}^2} = \mathrm{P}_1+\mathrm{P}_2 = 2\mathrm{K} \quad \text{say} \quad . \quad . \quad (19)$$

whence solving (14) and (19) for P_1 and P_2

$$\left.\begin{aligned} \mathrm{P}_1 &= \mathrm{K} + \left(\frac{\mathrm{L}}{k} - \mathrm{K}\right)\Big/\beta \\ \mathrm{P}_2 &= \mathrm{K} - \left(\frac{\mathrm{L}}{k} - \mathrm{K}\right)\Big/\beta \end{aligned}\right\} \quad . \quad . \quad . \quad (20)$$

and substituting these values in (15) we easily find

$$\beta = \frac{\mathrm{L}-\mathrm{K}k}{\sqrt{(\mathrm{L}-\mathrm{K}k)^2 + (\mathrm{M}k-\mathrm{L}^2)}} \quad . \quad . \quad . \quad (21)$$

We also find from (14) and (18)

$$\left.\begin{aligned} \mathrm{P}_1 &= \frac{1}{k}\left(\mathrm{L} + \sqrt{\frac{1-\beta}{1+\beta}}\sqrt{\mathrm{M}k-\mathrm{L}^2}\right) \\ \mathrm{P}_2 &= \frac{1}{k}\left(\mathrm{L} - \sqrt{\frac{1+\beta}{1-\beta}}\sqrt{\mathrm{M}k-\mathrm{L}^2}\right) \end{aligned}\right\} \quad . \quad . \quad (22)$$

Equations (21) and (22) constitute the solution of the problem.

The argument of P_1 is θ_1, and hv_1 can be found from the modulus by means of Table I. ζ_1 and ζ_2 can be taken from the same table, and then α is found from the formula

$$\frac{1-\alpha}{1+\alpha} = \frac{1-\beta}{1+\beta}\cdot\frac{\zeta_1}{\zeta_2}.$$

In the practical application of these formulæ a difficulty arises from the fact that, whereas β is by definition a real quantity, the right-hand side of (21), which is derived from observation, is in general a complex quantity.

The reason is this: equations (14), (15), and (16) practically constitute six equations, for each of them has a real and an imaginary part, which must be satisfied separately. But there are only five unknowns. In the above solution the equations have been reconciled by introducing a sixth fictitious unknown, viz. the imaginary part of α. If, however, α and β are to be purely real, the six equations cannot be exactly satisfied. Now it is clear that, just as we were justified in rejecting equations derived from equating the higher harmonics, we should attach more weight to equations (14) and (15) than to (16). It seems reasonable to satisfy the former exactly, and to throw the discordance entirely on (16). In other words, the observed and theoretical distributions will be made to agree exactly as regards coefficients of $\sin\theta$, $\cos\theta$, $\sin 2\theta$, $\cos 2\theta$, in the corresponding Fourier series, the coefficients of $\sin 3\theta$ and $\cos 3\theta$ will be made to agree as well as possible, and all higher harmonics will, as before, be entirely disregarded. If this rule seems rather arbitrary, it must be remembered that the whole question is one of weighting the equations of condition, that the ideal system of weighting is impracticable, and that we are seeking a practicable system which shall not unduly waste the material afforded by observation.

As (16) is not to be exactly satisfied, our solution must be chosen to make the sum of the squares of the two residuals (of the real and imaginary parts of the equation) a minimum. This is equivalent to making a minimum the modulus of the residual of the complex equation. Or we may satisfy the equation exactly by replacing N by $N + \delta N$, where $|\delta N|$ is to be a minimum.

Following this change through the solution previously given, let the alteration in N change $L - Kk$ to $(L - Kk)_0$; we see from (19) that if $|\delta N|$ is a minimum,

$$|(L - Kk) - (L - Kk)_0| \text{ will be a minimum.}$$

But (21) may be written

$$\frac{1}{\beta^2} = 1 + \left\{ \frac{\sqrt{Mk - L^2}}{L - Kk} \right\}^2;$$

therefore if β is real, we must choose $(L - Kk)_0$ so that its argument is the same as, or differs by 180° from, that of $\sqrt{Mk - L^2}$.

(The alternative $\arg (L - Kk)_0 = \arg \sqrt{Mk - L^2} \pm 90°$ leads to a value of β greater than unity, which must evidently be excluded.)

The conditions are satisfied if $(L - Kk)_0$ is the projection of the vector $L - Kk$ (in an Argand diagram) on the direction of the vector $\pm\sqrt{Mk - L^2}$; for this makes $|L - Kk - (L - Kk)_0|$ a minimum, subject to β being real.

Therefore

$$|(\mathrm{L}-\mathrm{K}k)_0| = |\mathrm{L}-\mathrm{K}k| \cos\{\arg(\mathrm{L}-\mathrm{K}k) - \arg\sqrt{\mathrm{M}k-\mathrm{L}^2}\} \quad (23)$$

and $$\frac{1}{\beta^2} = 1 + \frac{|\mathrm{M}k-\mathrm{L}^2|}{|(\mathrm{L}-\mathrm{K}k)_0|^2} \quad . \quad . \quad . \quad . \quad (24)$$

and P_1 and P_2 are given by (22) without modification.

The sign of β is positive or negative according as $(\mathrm{L}-\mathrm{K}k)_0$ is of the same or opposite sign to $\sqrt{\mathrm{M}k-\mathrm{L}^2}$. Either square root of $\mathrm{M}k-\mathrm{L}^2$ may be taken, but the same root must be adhered to throughout. It is, however, desirable to choose the root whose argument agrees roughly with the direction of motion of Drift I relative to Drift II, otherwise the suffixes of the constants will not agree with the usual designations of the drifts.

Example.

The following example shows the practical application of the method. It refers to Region C (R.A. 2^h–6^h, N.P.D. 20°–52°) of the Groombridge proper motions.

From the observed proper motions I find

Total number of stars $n = 512$.

$$\Sigma\cos\theta = +73{\cdot}0 \qquad \Sigma\sin\theta = +219{\cdot}0$$
$$\Sigma\cos 2\theta = -42{\cdot}3 \qquad \Sigma\sin 2\theta = +124{\cdot}1$$
$$\Sigma\cos 3\theta = -98{\cdot}45 \qquad \Sigma\sin 3\theta = +\ 23{\cdot}1$$

hence, dividing by n,

$$\mathrm{L} = +{\cdot}143 + {\cdot}428\,\iota$$
$$\mathrm{M} = -{\cdot}083 + {\cdot}242\,\iota$$
$$\mathrm{N} = -{\cdot}192 + {\cdot}045\,\iota$$
and $$\gamma\mathrm{N} = -{\cdot}223 + {\cdot}052\,\iota \qquad (\gamma = 1{\cdot}163).$$

Assume provisionally $k = 1{\cdot}40$, we find

$$\mathrm{N}\gamma k - \mathrm{LM} = -{\cdot}1967 + {\cdot}0737\iota = {\cdot}2100 \text{ expi* } 159^\circ{\cdot}5$$
$$\mathrm{M}k - \mathrm{L}^2 = +{\cdot}0466 + {\cdot}2164\iota = {\cdot}2214 \text{ expi } 77^\circ{\cdot}9$$

therefore by division (equation (19))

$$2\mathrm{K} = {\cdot}948 \text{ expi } 81^\circ{\cdot}6$$
$$\mathrm{K} = +{\cdot}069 + {\cdot}469\,\iota$$

therefore $$\mathrm{L}-\mathrm{K}k = +{\cdot}046 - {\cdot}229\,\iota$$
$$= {\cdot}234 \text{ expi } 281^\circ{\cdot}4\,;$$

also $$\sqrt{\mathrm{M}k-\mathrm{L}^2} = {\cdot}470 \text{ expi } 39^\circ{\cdot}0.$$

If we continue the solution with $\mathrm{L}-\mathrm{K}k$ unmodified, equation (21) gives

$$\beta = -{\cdot}300 - {\cdot}442\,\iota.$$

* Expi $x = e^{\iota x}$.

But modifying it in accordance with (23)

$$(\mathrm{L}-\mathrm{K}k)_0 = \cdot 234 \cos(281^\circ\cdot 4 - 39^\circ\cdot 0) \operatorname{expi} 39^\circ\cdot 0$$
$$= -\cdot 1083 \operatorname{expi} 39^\circ\cdot 0$$

and
$$\frac{\sqrt{\mathrm{M}k-\mathrm{L}^2}}{(\mathrm{L}-\mathrm{K}k)_0} = -4\cdot 53,$$

whence
$$\beta = -\frac{1}{\sqrt{1+(4\cdot 53)^2}} = -\cdot 215.$$

This value of β must now be substituted in (22),

$$\sqrt{\frac{1-\beta}{1+\beta}} = 1\cdot 244.$$

$$\sqrt{\frac{1-\beta}{1+\beta}}\sqrt{\mathrm{M}k-\mathrm{L}^2} = \cdot 610 \operatorname{expi} 39^\circ\cdot 0$$
$$= \cdot 474 + \cdot 384\,\iota$$

and $$\mathrm{L} = \cdot 143 + \cdot 428\,\iota.$$

Sum $$= k\mathrm{P}_1 = \cdot 617 + \cdot 812\,\iota,$$
$$= 1\cdot 020 \operatorname{expi} 52^\circ\cdot 8.$$

Therefore $$\mathrm{P}_1 = \cdot 729 \operatorname{expi} 52^\circ\cdot 8.$$

Similarly we find $$\mathrm{P}_2 = \cdot 174 \operatorname{expi} 132^\circ\cdot 7.$$

The directions of the two drifts are accordingly

$$\theta_1 = 52^\circ\cdot 8 \qquad \theta_2 = 132^\circ\cdot 7.$$

To find their velocities we make use of Table I., which gives the corresponding values of $|\mathrm{P}|$ and hv,

$$hv_1 = 1\cdot 61 \qquad hv_2 = 0\cdot 31\,;$$

also from Table I.,

$$\zeta_1 = 1\cdot 211 \qquad \zeta_2 = 1\cdot 547$$

but
$$\frac{1-\alpha}{1+\alpha} = \frac{1-\beta}{1+\beta}\frac{\zeta_1}{\zeta_2}$$
$$= 1\cdot 548 \times \frac{1\cdot 211}{1\cdot 547}$$
$$= 1\cdot 212,$$

whence $$\alpha = -\cdot 096$$

$$\tfrac{1}{2}(1+\alpha) = \cdot 452 \quad \tfrac{1}{2}(1-\alpha) = \cdot 548.$$

Thus 45·2 per cent. of the stars belong to Drift I and 54·8 per cent. to Drift II.

Finally, we can check the provisional value of k,

$$k = \tfrac{1}{2}(1+\alpha)\zeta_1 + \tfrac{1}{2}(1-\alpha)\zeta_2 = 1\cdot 395,$$

agreeing with the adopted value 1·40, so that no further approximation is needed.

The results obtained by this method agree closely with those previously obtained by the method of trial and error ("Systematic Motions," p. 50). I give below a comparison of the constants determined in the two ways for four of the Groombridge Regions; it affords some sort of indication of the reliability of both methods.

		Analytical Method.		Trial and Error.	
		Drift I.	Drift II.	Drift I.	Drift II.
Region A	hv	1·38	0·31	1·65	0·30
	θ	0°	157°	0°	155°
	n*	52·5	47·5	48·5	51·5
Region B	hv	1·47	0·48	1·55	0·45
	θ	12°	131°	10°	130°
	n	49·5	50·5	49·5	50·5
Region C	hv	1·61	0·31	1·65	0·30
	θ	53°	133°	55°	125°
	n	45·2	54·8	41·5	58·5
Region F	hv	0·86	0·77	1·20	0·45
	θ	226°	75°	225°	80°
	n	63·0	37·0	46·5	53·5

* Per cent.

The weakest point in the solution is exemplified in the case of Region F; the determination of α (or β) is not very satisfactory. The difference in the two solutions in the case of Region F depends entirely on the division of the stars between the two drifts; if we had adopted the same value of α in the two cases, the remaining four constants found from (22) would have agreed almost exactly with those found by trial and error. The weakness of the determination of β can be shown analytically.

Differentiating (21),

$$\frac{d\beta}{dN} = -\tfrac{1}{2}k^2\gamma\{(Mk - L^2) + (L - Kk)^2\}^{-\frac{3}{2}}$$

$$= -\frac{4\gamma}{k}(P_1 - P_2)^{-3}.$$

Now $|P_1 - P_2|$ is generally about 0·7, but may be less, hence $\left|\frac{d\beta}{dN}\right|$ is generally about 10, but may be more.

Thus in a region containing 500 stars, a change of 2·5 in $\Sigma \cos 3\theta$ or $\Sigma \sin 3\theta$ would produce a change in N of ·005, and the corresponding change in β or α might be $\pm$·05.

The question arises whether the weakness of the determination of β is a defect of the method of analysis or is necessarily involved in the nature of the observational data. I think there is little doubt that the latter alternative is correct; I have examined various other methods of determining β, but all are rather insensitive. In the case of Region F, both the solutions given above are found to agree with the observed distributions almost equally well.

Thus it seems likely that even in an ideal solution we should have one equation very much weaker than the other four (the weakness may be more pronounced for some regions of the sky than for others). As the five constants depend on one another and on this fifth equation, they may all share in the uncertainty.

One or two considerations help to avoid this difficulty to some extent. We may be content to assume $a=0$, *i.e.* that the stars are evenly divided between the two drifts; all evidence seems to indicate that this is approximately true, and it is conceivable that there may be some physical reason for it. Or, instead of adopting the value of a found for the particular region, we may adopt a mean found from all the regions discussed; this will have a much smaller probable error. But the most fortunate circumstance is that *we may determine the relative motion of the two drifts almost independently of* a.

Equation (18) gives

$$P_1 - P_2 = \frac{2}{k}(1-\beta^2)^{-\frac{1}{2}}\sqrt{Mk - L^2}\,.$$

If β lies anywhere between $-0{\cdot}3$ and $+0{\cdot}3$, $(1-\beta^2)^{-\frac{1}{2}}$ may be put equal to $\cdot 98$ with an error certainly less than 3 per cent.; but this range of β includes all values likely to occur. Only in the case of a very great disparity in the distribution between the two drifts could a value outside these limits occur. Thus $P_1 - P_2$ is nearly independent of β or a. P is a sufficiently nearly linear function of hV for $h(V_1 - V_2)$, to be also nearly independent of a. Thus although adopting $a=0$ may lead to some error in the determinations of hV_1 and hV_2, the error will nearly be eliminated from the determination $h(V_1 - V_2)$. This relative motion of the two drifts is the quantity which most interests us, especially as a systematic error in the proper motions does not affect its determination so adversely as it affects the determinations of hV_1 and hV_2.

Systematic Motions of Zodiacal Stars.

I have applied the theory given above to the discussion of the proper motions of the zodiacal stars. The proper motions were taken from the Catalogue of Zodiacal Stars, *Astronomical Papers of the American Ephemeris*, vol. viii., part iii. Excluding the stars of the Pleiades, this contains 1533 proper motions. I divided the zodiac into sixteen regions, each extending $22\frac{1}{2}°$ in longitude by about 16° in latitude; these are denoted successively by I*a*, II*a*, ... VIII*a*, I*b*, ... VIII*b*, the centre of region I*a* being at the first point of Aries. As regions I*a* and I*b* are diametrically opposite to one another, the observed motions are in parallel planes, and the two regions may be treated together; similarly, the other regions can be treated together in pairs. Thus the number of regions is virtually reduced to eight, each containing from 150 to 250 stars.

The distribution of the proper motions as regards direction in the eight regions is shown in Table II. Opposite $\theta = 0°$ in the

first column is given the number of stars whose observed motions are in directions between $\theta=355^\circ$ and $\theta=5^\circ$, and so on. The numbers have not been smoothed. For stars in I*a*, II*a*, etc., $\theta=0^\circ$ is in the direction of increasing R.A., and $\theta=90^\circ$ in the direction of increasing Dec. For stars in I*b*, II*b*, etc., the reckoning of θ agrees with that in the opposite parallel planes I*a*, II*a*, etc.

TABLE II.

Distribution of the Proper Motions in Direction.

	Regions.							
θ	I.	II.	III.	IV.	V.	VI.	VII.	VIII.
0°	16	14	9	5	7	2	1	2
10	11	8	3	8	0	0	5	4
20	3	3	3	2	1	1	2	2
30	2	6	0	2	3	4	2	1
40	1	2	2	0	1	2	0	2
50	4	3	0	2	3	2	3	0
60	2	1	0	1	0	1	1	0
70	3	2	0	0	0	0	2	2
80	2	1	0	1	1	0	0	2
90	3	3	1	1	1	0	1	1
100	1	1	1	0	1	1	0	0
110	1	1	0	2	1	1	2	0
120	0	2	1	1	0	4	2	3
130	0	1	1	2	3	2	2	4
140	1	0	1	2	0	3	5	4
150	2	1	0	1	4	1	3	5
160	3	2	1	2	5	6	4	3
170	1	2	3	2	3	2	9	11
180	0	1	2	2	6	10	8	13
190	6	4	1	1	6	7	18	14
200	6	8	4	3	8	9	12	12
210	5	5	6	3	5	9	12	14
220	6	4	3	5	11	16	14	8
230	10	5	6	8	11	20	10	6
240	2	7	2	4	13	14	16	4
250	2	6	3	10	9	17	13	4
260	1	3	6	7	9	9	9	5
270	6	5	2	7	21	12	3	8
280	3	6	9	12	17	10	6	4
290	4	7	13	20	10	8	10	4
300	6	7	11	18	16	8	0	10
310	12	7	8	13	13	6	7	3
320	17	11	15	27	13	6	2	2
330	14	4	12	18	8	4	2	5
340	9	20	18	23	6	1	0	3
350	10	17	9	25	5	0	8	4
Total	175	180	156	240	221	198	194	169

It will be seen, by looking down the columns of the Table that the two streams are plainly marked in Regions I, II, III, and VIII. In the other four regions their directions (projected on the sky) are inclined at an acute angle, and the existence of the two streams is rather concealed. The success of the analysis in these four regions is on that account especially interesting. It may be noticed that this belt of the sky is not so favourable as the Groombridge region for showing conspicuously the separation into two streams; the centre of the latter region lies between the two apices, so that in it the streams are in nearly opposite directions.

Although, in the main, pairs of regions such as I*a* and I*b* were treated together, I thought it safer to calculate $\sqrt{Mk - L^2}$ and L for I*a* and I*b* separately, and to take the mean afterwards. This was in order to avoid the possible effects of systematic error, by ensuring that the difference of motion of the two drifts found from the observations was a difference of motion of intermingled systems of stars, and not the difference in the apparent motion of stars in I*a* from that of stars in I*b*. Actually, however, the precaution might have been omitted. I found that in every region very nearly the same result was obtained whether the two halves were treated separately or together.

For the solutions I used entirely the equations—

$$P_1 = \frac{1}{k}\left(L + \sqrt{\frac{1-\beta}{1+\beta}}\sqrt{Mk - L^2}\right)$$

$$P_2 = \frac{1}{k}\left(L - \sqrt{\frac{1+\beta}{1-\beta}}\sqrt{Mk - L^2}\right)$$

I did not calculate N, or attempt to find β from the observations. The final results given below depend, therefore, on the assumption (based on previous experience) that the stars are evenly distributed between the two drifts; but calculations are given which show to what extent the results obtained would need to be modified if the assumption is incorrect.

As a preliminary I made two solutions, (*a*) assuming $\sqrt{\frac{1-\beta}{1+\beta}} = 1{\cdot}0$ and (*b*) assuming $\sqrt{\frac{1-\beta}{1+\beta}} = 1{\cdot}1$; in both cases k was assumed to be 1·40. These correspond to assuming that the stars belonging to Drift I are about (*a*) 55 per cent., (*b*) 50 per cent. of the whole.

The combined results from all the regions were—

Velocity of Drift I	(*a*)	1·61	towards Latitude	−36°,	Longitude	105°
	(*b*)	1·74	,,	−34°,	,,	106°
Velocity of Drift II	(*a*)	0·64	,,	−42°,	,,	302°
	(*b*)	0·60	,,	−48°,	,,	304°
Velocity of Drift I relative to Drift II	(*a*)	1·84	,,	−16°·3,	,,	109°·4
	(*b*)	1·90	,,	−16°·5,	,,	109°·6

The unit of velocity is $1/h$.

The mutual relative velocity of the two drifts is thus determined nearly independently of the assumed division of the stars between them, a result which has already been arrived at theoretically.

For a final solution, using (*b*) as a first approximation, I calculated for each region values of k and β, assuming that the stars were equally divided between the drifts.

The values of k for the Regions I, II, ... VIII were respectively—

1·35, 1·35, 1·36, 1·40, 1·44, 1·44, 1·43, 1·38,

and of $\sqrt{\dfrac{1-\beta}{1+\beta}}$

1·12, 1·12, 1·11, 1·09, 1·07, 1·07, 1·08, 1·10.

Performing the analysis with these values, the constants of the drifts in the eight regions were found as follows:—

Region.	Longitude of Centre.	Drift I.		Drift II.	
		hv_1.	θ_1.	hv_2.	θ_2.
I	0° , 180°	1·57	345°·4	0·46	244°·8
II	22½ , 202½	1·46	350 ·0	0·58	249 ·4
III	45 , 225	1·86	330 ·0	0·63	252 ·2
IV	67½ , 247½	1·68	323 ·5	0·51	271 ·3
V	90 , 270	1·08	299 ·7	0·70	231 ·2
VI	112½ , 292½	1·04	245 ·4	0·54	233 ·3
VII	135 , 315	1·35	208 ·8	0·40	275 ·8
VIII	157½ , 337½	1·49	189 ·2	0·54	292 ·3

If we resolve these drift-velocities along and perpendicular to the plane of the ecliptic, the components perpendicular to the ecliptic derived from each region should agree.

The determinations of these components are—

for Drift I, − ·97, − ·77, − 1·35, − 1·19, − ·94, − 1·00, − ·96, − ·76, Mean − ·99
for Drift II, − ·30, − ·43, − ·52, − ·51, − ·54, − ·48, − ·37, − ·39, Mean − ·44

The accordance of these seems decidedly good when it is remembered that they are derived from regions containing on an average less than 200 stars.

The component drift-velocities in the ecliptic are more or less foreshortened according to the longitude of the region; I determined the mean values by a least-squares solution.

The final results are—

Velocity of Drift I,	1·78	towards	Latitude	−33°·8,	Longitude	105°·5
Velocity of Drift II,	0·59	,,	,,	−47°·9,	,,	303°·7
Velocity of Drift I, relative to Drift II,	1·94	,,	,,	−16°·4,	,,	109°·3

The slight difference between this result and that of solution (*b*) represents the effect of allowing for the variation of β and k from region to region, instead of adopting a mean. The comparison between solutions (*a*) and (*b*) serves to indicate how the results would be changed if the stars were not evenly divided between the drifts. The constants most affected by such a change are the speed of Drift I and the direction of Drift II; the other constants are nearly independent of the assumption, and are therefore more reliably determined.*

From an examination of residuals, I estimate that the probable accidental error of the determination of hV_1 and hV_2 is about $\pm\cdot06$, and the probable errors of the apices of Drifts I and II are respectively about 2° and 6° of a great circle.

Converting latitude and longitude into R.A. and Dec., the positions of the antapices may be compared with previous determinations as follows—

	Drift I.		Drift II.	
Kapteyn	R.A. 85°	Dec. −11°	R.A. 260°	Dec. −48°
Dyson	,, 94°	,, −7°	,, 240°	,, −74°
Groombridge stars .	,, 90°	,, −19°	,, 292°	,, −58°
Zodiacal stars . . .	,, 103°	,, −11°	,, 330°	,, −64°

The great R.A. of the antapices of both drifts found in the present discussion is rather hard to account for.

The velocities of the two drifts $hv_1 = 1\cdot78$, $hv_2 = 0\cdot59$ are in excellent agreement with those found from the Groombridge stars $hv_1 = 1\cdot7$, $hv_2 = 0\cdot5$.

For the velocity of one drift relative to the other, the Zodiacal stars give the value 1·94. From the Groombridge stars (by a least-squares solution from the results of the separate regions) I have found the value 1·90. The determinations of the point towards which this relative velocity is directed (called by Professor Kapteyn the *true* vertex) are

Kapteyn	R.A. 91°	Dec. +13°
From Groombridge stars .	,, 95°	,, +3°
,, Zodiacal ,, .	,, 109°	,, +6°

Professor Schwarzschild's determination of the line of symmetry of motion may be added. This, although based on a rather different theory, is directly comparable with the above. He found,

From Groombridge stars . . R.A. 93° Dec. +6°

* It should be understood that it is not impossible to determine α from the observations, but simply that when, as in the present case, a few stars are discussed, the value of α is liable to a greater uncertainty than some of the other results. Undoubtedly from the whole 1533 stars a fairly good mean value of α could be determined; but the task of computing it for sixteen regions separately, and taking the mean, would be laborious.

Further, from Professor Dyson's investigation, an R.A. of about 92° for this point may be inferred.

Thus the Right Ascension given by the Zodiacal proper motions is discordant as compared with the other determinations. The number of stars here considered is fewer, and the proper motions are perhaps not so well determined, but I do not think the discordance can be altogether attributed to this. Nor can it be traced to a local anomaly, for it seems to be indicated by the proper motions all round the ecliptic. I have verified by calculation that the Regions I*a* to VIII*a* agree with Regions I*b* to VIII*b* in leading to this high value of the Right Ascension.

Note on the evaluation of certain integrals required in the analysis.

(1) To calculate $n_1 C_1 = \int_0^{2\pi} \rho \cos(\theta - \theta_1) d\theta$, for a single drift.

We may choose the initial line, along the direction of the drift, so that $\theta_1 = 0$.

Then
$$\rho = \frac{n_1}{\pi} e^{-h^2 v_1^2} \left\{ \tfrac{1}{2} + \tau e^{\tau^2} \int_{-\tau}^{\infty} e^{-x^2} dx \right\}$$

where $\tau = h v_1 \cos\theta$.

Now $\left\{ \tfrac{1}{2} + \tau e^{\tau^2} \int_{-\tau}^{0} e^{-x^2} dx \right\} \cos\theta$ is an odd function of $\cos\theta$ and vanishes when integrated from 0 to 2π.

Hence
$$\int_0^{2\pi} \rho \cos\theta d\theta = \frac{n_1}{\pi} e^{-h^2 v_1^2} \frac{\sqrt{\pi}}{2} \int_0^{2\pi} \tau e^{\tau^2} \cos\theta d\theta$$
$$= \frac{n_1 h v_1}{2\sqrt{\pi}} e^{-h^2 v_1^2} \int_0^{2\pi} \cos^2\theta e^{h^2 v_1^2 \cos^2\theta} d\theta.$$

Now
$$\int_0^{2\pi} e^{2\kappa \cos^2\theta} d\theta = e^{\kappa} \int_0^{2\pi} e^{\kappa \cos 2\theta} d\theta$$
$$= 2\pi e^{\kappa} J_0(\iota\kappa).$$

Whence differentiating with respect to κ,
$$2\int_0^{2\pi} \cos^2\theta e^{2\kappa \cos^2\theta} d\theta = 2\pi e^{\kappa}\big(J_0(\iota\kappa) - \iota J_1(\iota\kappa)\big).$$

Now write $\kappa = \tfrac{1}{2} h^2 v_1^2$ and substitute above
$$C_1 = \frac{\sqrt{\pi}}{2} h v_1 e^{-\frac{1}{2} h^2 v_1^2} \left\{ J_0(\iota \tfrac{1}{2} h^2 v_1^2) - \iota J_1(\iota \tfrac{1}{2} h^2 v_1^2) \right\}.$$

Tables of the Bessel Functions of an imaginary argument are given in *Brit. Assoc. Reports*, 1893 and 1896, and have been used in calculating Table I.

(2) To calculate $n_1 D_1 = \int_0^{2\pi} \rho \cos 2\theta d\theta.$

We have $\rho = \frac{n_1 h^2}{\pi} \int_0^\infty e^{-h^2(r^2+v_1^2-2v_1 r\cos\theta)} r dr$

and identically

$$n_1 = \int_0^{2\pi} \rho d\theta = \frac{n_1 h^2}{\pi} \int_0^{2\pi} d\theta \int_0^\infty e^{-h^2(r^2+v_1^2-2v_1 r\cos\theta)} r dr$$

$$= \frac{n_1 h^2}{\pi} \int_0^\infty e^{-h^2(r^2+v_1^2)} r dr \int_0^{2\pi} e^{z\cos\theta} d\theta . \quad . \quad . \quad . \quad (1)$$

where $z = 2h^2 v_1 r$.

Hence $$n_1 \equiv \frac{n_1 h^2}{\pi} \int_0^\infty e^{-h^2(r^2+v_1^2)} r dr . 2\pi J_0(\iota z) . \qquad (2)$$

But from (1)

$$\int_0^{2\pi} \rho \cos 2\theta d\theta = \frac{n_1 h^2}{\pi} \int_0^\infty e^{-h^2(r^2+v_1^2)} r dr \frac{d^2}{dz^2} \int_0^{2\pi} e^{z\cos\theta} d\theta$$

$$= \frac{n_1 h^2}{\pi} \int_0^\infty e^{-h^2(r^2+v_1^2)} r dr \frac{d^2}{dz^2} \left(2\pi J_0(\iota z) \right)$$

$$= \frac{n_1 h^2}{\pi} \int_0^\infty e^{-h^2(r^2+v_1^2)} r dr \left(2\pi J_0(\iota z) - \frac{1}{z} \frac{d}{dz} \left(2\pi J_0(\iota z) \right) \right)$$

by Bessel's differential equation

$$= n_1 - \frac{n_1 h^2}{\pi} \int_0^\infty e^{-h^2(r^2+v_1^2)} \frac{1}{4h^4 v_1^2} \frac{d}{dr} \left(2\pi J_0(\iota z) \right) dr$$

Integrate by parts:

$$= n_1 - \frac{n_1 h^2}{\pi} \frac{1}{4h^4 v_1^2} \left[e^{-h^2(r^2+v_1^2)} 2\pi J_0(\iota 2h^2 v_1 r) \right]_0^\infty$$

$$+ \frac{n_1 h^2}{\pi} \frac{1}{4h^4 v_1^2} \int_0^\infty dr(-2h^2 r) e^{-h^2(r^2+v_1^2)} 2\pi J_0(\iota z)$$

$$= n_1 + \frac{n_1}{2h^2 v_1^2} e^{-h^2 v_1^2} - \frac{n_1}{2h^2 v_1^2}$$

(the integrated part vanishes for $r = \infty$ and reduces to $\frac{n}{2h^2 v_1^2} e^{-h^2 v_1^2}$

at the lower limit; the part remaining to be integrated is simplified by means of the identity (2)),

$$\text{hence} \qquad \int_0^{2\pi} \rho \cos 2\theta d\theta = 2\int_0^{2\pi} \rho \cos^2\theta d\theta - \int_0^{2\pi} \rho d\theta$$
$$= n_1\left(1 - \frac{1 - e^{-h^2 v_1^2}}{h^2 v_1^2}\right).$$

(3) The integral $\int_0^{2\pi} \rho \cos 3\theta d\theta$, and corresponding integrals for any odd multiples of θ, can be found by a simple extension of the method employed for $\int_0^{2\pi} \rho \cos \theta d\theta$. The integrals for even multiples of θ are more troublesome to evaluate, but the method employed for $\int_0^{2\pi} \rho \cos 2\theta d\theta$ always succeeds.

Tables of the two hypergeometrical functions, $F\left(1/6, 5/6, 2, \sin^2 \frac{\iota}{2}\right)$ *and* $F\left(-1/6, 7/6, 2, \sin^2 \frac{\iota}{2}\right)$, *between the limits iota equals* 90 *and* 180 *degrees.* By C. J. Merfield.

In the method of Mr. R. T. A. Innes for the determination of the secular perturbations,* there are two hypergeometrical functions to be deduced. In an appendix † to this valuable paper, tables of the logarithms of these functions are given with the argument iota for each degree ‡ for the first quadrant.

Tables of these functions facilitate the application of this method in no small degree, and it seemed desirable to extend them, as in many future investigations it will be found that the angle iota will much exceed a quadrant. Taking an example, Eros §—Earth, it will be noted that the modular angle theta, the argument to the tables of elliptical integrals, reaches the value 60°, corresponding to iota 138° 18′, and there will be many other cases in which it exceeds this value.

In the preparation of the tables here given the formulæ || (*l.c.*,

* "Computation of Secular Perturbations," by R. T. A. Innes, *Monthly Notices*, vol. lxvii. 427.

† Tables for the application of Mr. Innes's Method, by Frank Robbins. *l.c.*, 444.

‡ The values of these functions for iota equals 0° have been omitted in the tabulation by Mr. Robbins.

§ "Secular Perturbations of Eros," by C. J. Merfield, *Astr. Nachr.*, 4178–79, Band 175.

|| The values of these functions may be deduced from series. I have given the coefficients of twenty terms, *Astr. Nachr.*, 4215, Band 176, p. 246.